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Comparison of Flexural Behaviour of Functionally Graded Plates Using Higher Order Shear Deformation Theory with Different Homogenization Schemes

Sandra Mohan P, Dr Shashidharan

Abstract—Functionally graded materials are one of the most widely used materials in various applications because of their adaptability to different situations by changing the material constituents as per the requirement. The properties of functionally graded plates can be defined by using different homogenization methods like Simple power law distribution, rule of mixtures, Mori-Tanaka homogenization scheme, sigmoid function exponential distribution function etc. In the present study the power law, Mori-Tanaka scheme and exponential distribution is considered for the volume fraction distribution of functionally graded plates. The flexural behavior is studied for different homogenization schemes and obtained results are compared with the available literatures. The basic formulations for the FGM plate are done based on the third order shear deformation theory. All the numerical examples are done using MATLAB programs.

Index Terms— Functionally Geaded Plates, Different Homogenization Schemes, MATLAB

1 INTRODUCTION

THE functionally graded plates are the latest revolutionary engineering material in which the material properties are continually varied through the thickness direction by mixing two different materials and thus no distinct internal boundaries exist and failures from interfacial stress concentrations developed in conventional structural components can be avoided. FGMs are widely used in many structural applications such as mechanics, civil engineering, optical, electronic, chemical, mechanical, biomedical, energy sources, nuclear, automotive fields and ship building industries to eliminate stress concentration and relax residual stresses and enhance bond strength.

The continuous change in the microstructure of Functionally Graded Materials (FGMs) distinguish them from the fiber-reinforced laminated composite materials, which have a mismatch of mechanical properties across aninterface due to two discrete materials bonded together. As a result, the constituents of the fiber matrix composites are prone to de-bonding at extremely high thermal loading. Further, cracks are likely to initiate at the interfaces and grow into weaker material sections. Additional problems include the presence of residual stresses due to the difference in coefficients of thermal expansion of the fiber and matrix in the composite materials. In FGMs these problems are avoided or reduced by gradually varying the volume fraction of the constituents rather than abruptly changing them across an interface.

- *Sandra Mohan P is currently pursuing masters degree program in Structural Engineering in NSS College of Engineering, Palakkad. E-mail: sandramohanp@gmail.com*
- *Dr. Shashidharan is currently working as Professor in Department of Civil Engineeringineering in NSS College of Engineering,Palakkad. E-mail:shashisubha@gmail.com*

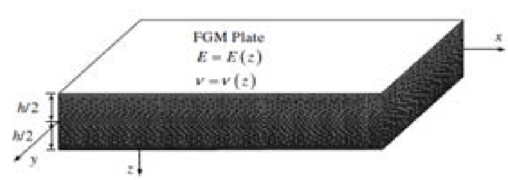

Fig. 1 Geometry of an FGM Plate

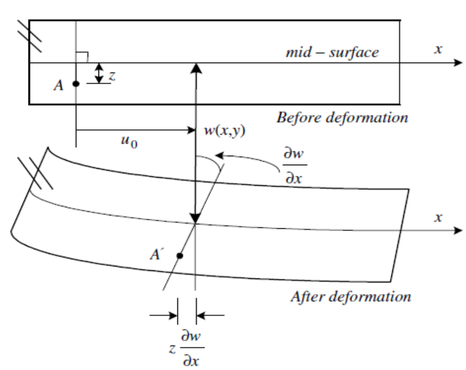

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Fig. 2 Deformed Configuration of an FGM Plate

2 Different Homogenization Schemes

2.1 Power law distribution function

Using simple power law distribution it is assumed that the material property gradation is through the thickness and we represent the profile for volume fraction by the expression

$$P(z) = (P_t - P_b)V + P_b$$

$$V = \left(\frac{z}{h} + \frac{1}{2}\right)^n \qquad (1)$$

where P denotes a generic material property like modulus, P_t and P_b denote the property of the top and bottom faces of the plate, respectively, h is the total thickness of the plate, and n is a parameter that dictates the material variation profile through the thickness.

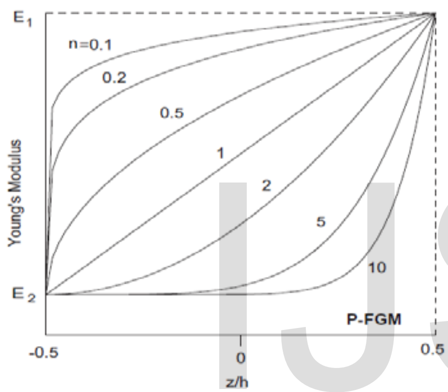

Fig. 3Vaiation of Young's Modulus in a P-FGM Plate

2.2 Mori-Tanaka homogenization scheme

Mori-Tanaka homogenization method can be used to the effective properties of functionally graded material square plate by following steps. The effective bulk modulus K and shear modulus G of functionally graded material evaluated using Mori-Tanaka estimates are as

$$\frac{K - K_m}{K_c - K_m} = \frac{V_c}{1 + (1 - V_c)\dfrac{3(K_c - K_m)}{(3K_m + 4G_m)}}$$

$$\frac{G - G_m}{G_c - G_m} = \frac{V_c}{1 + (1 - V_c)\dfrac{G_c - G_m}{(G_m + f_1)}}$$

$$f_1 = \frac{G_m(9K_m + 8G_m)}{6(K_m + 2G_m)} \qquad (2)$$

where

$$V_c(z) = \left(\frac{2z + h}{2h}\right)^k \qquad (3)$$

where$k$=volume fraction exponent (k>=0). The variation of the composition of ceramic and metal is linear for k=1. The value of k equal to zero represents a fully ceramic plate. The effective values of Young's modulus E and Poisson's ratio v can be found from $E=(9KG/3K+G)$ and v=(3K−2G) / 2(3K+G), respectively.

2.3 EXPONENTIAL DISTRIBUTION FUNCTION

The exponential function to describe the material properties of FGMs as follows

$$E(z) = Ae^{B(z+\frac{h}{2})} \qquad (4)$$

$$with$$

$$A = E_2$$

$$B = \frac{1}{h}\ln\left(\frac{E_1}{E_2}\right) \qquad (5)$$

3. THIRD ORDER SHEAR DEFORMATION THEORY

The flexural analysis of the functionally graded plates is done based on the formulations of third order shear deformation theory. The theory uses a displacement field as follows

$$u(x,y,t) = u_0(x,y,t) + z\Phi_x(x,y,t) - c_1 z^3\left(\Phi_x + \frac{\partial w_0}{\partial x}\right)$$

$$v(x,y,t) = v_0(x,y,t) + z\Phi_y(x,y,t) - c_1 z^3\left(\Phi_y + \frac{\partial w_0}{\partial y}\right)$$

$$w(x,y,t) = w_0(x,y,t) \qquad (6)$$

$$\begin{Bmatrix}\sigma_x \\ \sigma_y \\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{Bmatrix}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} Q_{44} & 0 \\ 0 & Q_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \quad (7)$$

The equilibrium equations are

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x}(N_{xx}\frac{\partial w_0}{\partial x} + N_{xy}\frac{\partial w_0}{\partial y}) + \frac{\partial}{\partial y}(N_{xy}\frac{\partial w_0}{\partial x} + N_{yy}\frac{\partial w_0}{\partial y}) +$$

$$C1(\frac{\partial^2 P_{xx}}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_{yy}}{\partial y^2}) + q = 0$$

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y = 0 \quad (8)$$

Where the stress resultant are defined as

$$\left(N_x, N_y, N_{xy}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left(\sigma_x, \sigma_y, \tau_{xy}\right) dz$$

$$\left(M_x, M_y, M_{xy}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left(\sigma_x, \sigma_y, \tau_{xy}\right) z dz$$

$$\left(P_x, P_y, P_{xy}\right) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left(\sigma_x, \sigma_y, \tau_{xy}\right) z^3 dz$$

$$\left(Q_x, Q_y\right) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\sigma_{yz}, \sigma_{xz}\right) dz$$

$$\left(R_x, R_y\right) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\sigma_{yz}, \sigma_{xz}\right) z^2 dz \quad (9)$$

Stress resultants are related to the strain by relations

$$\begin{Bmatrix} N \\ M \\ P \end{Bmatrix} = \begin{bmatrix} A & B & E \\ B & D & F \\ E & F & H \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \varepsilon^1 \\ \varepsilon^3 \end{Bmatrix}$$

$$\begin{pmatrix} Q \\ R \end{pmatrix} = \begin{bmatrix} A & D \\ D & F \end{bmatrix} \begin{Bmatrix} \gamma^0 \\ \gamma^2 \end{Bmatrix}$$

$$\left(A_{ij}, B_{ij}, D_{ij}, E_{ij}, H_{ij}\right) = \int_{-h/2}^{h/2} (\overline{Q_{ij}})_k (1, z, z^2, z^3, z^4, z^6) dz$$

$$\left(A_{ij}, D_{ij}, F_{ij}\right) = \int_{-h/2}^{h/2} (\overline{Q_{ij}})_k (1, z^2, z^6) dz \quad (10)$$

Where A_{ij} are called extensional stiffness, B_{ij} the bending extensional coupling stiffness and D_{ij} the bending stiffness and E_{ij}, F_{ij}, H_{ij} are stiffness associated with the transverse shear effects.

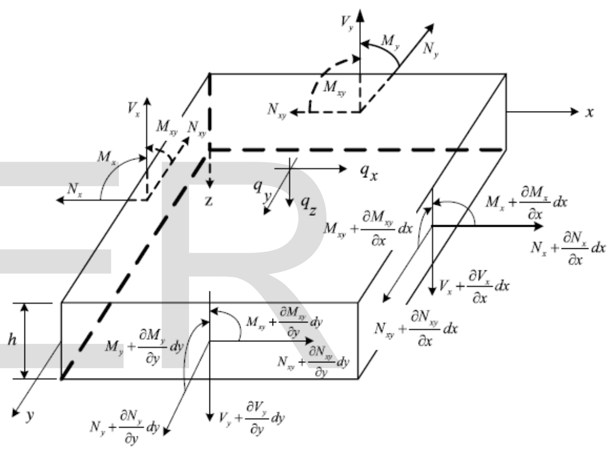

Fig. 4 Forces in a small element of an FGM Plate

To obtain deflections for FGP, SS-1 boundary conditions are used(Navier Solution). The boundary conditions are
At x=0 and x=a

$$v_0 = w_0 = \Phi_y = N_x = M_x = S_x = 0$$

At y=0 and y=b

$$u_0 = w_0 = \Phi_x = N_y = M_y = S_y = 0$$

The displacement boundary conditions are satisfied by assuming the following shape of displacements

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$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\Phi_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\Phi_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \quad (11)$$

The transverse load q is also expanded in double Fourier series as

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (12)$$

Substituting the displacement quantities in the governing equations we get

[C]{Δ}={F}

$$\{\Delta\} = \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} \text{ and } \{F\} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} \quad (13)$$

And the elements of symmetric matrix [C] is given as

$$C_{22} = A_{66}\alpha^2 + A_{22}\beta^2$$

$$C_{23} = -c1(E_{22}\beta^2 + (E_{12} + 2E_{66})\alpha^2)\beta$$

$$C_{24} = C_{15}$$

$$C_{25} = \hat{B}_{66}\alpha^2 + \hat{B}_{22}\beta^2$$

$$C_{33} = \overline{A}_{55}\alpha^2 + \overline{A}_{44}\beta^2 + c_1^{\,2}(H_{11}\alpha^4 + 2H_{12} + 2H_{66})\alpha^2\beta^2 + H_{22}\beta^4)$$

$$C_{34} = \overline{A}_{55}\alpha - c_1\hat{F}_{11}\alpha^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha\beta^2)$$

$$C_{35} = \overline{A}_{44}\beta - c_1\hat{F}_{22}\beta^3 + (\hat{F}_{12} + 2\hat{F}_{66})\alpha^2\beta)$$

$$C_{44} = \overline{A}_{55} + \overline{D}_{11}\alpha^2 + \overline{D}_{66}\beta^2$$

$$C_{45} = (\overline{D}_{12} + \overline{D}_{66})\alpha\beta$$

$$C_{55} = \overline{A}_{44} + \overline{D}_{66}\alpha^2 + \overline{D}_{22}\beta^2 \quad (14)$$

Where

$$\alpha = \frac{m\pi}{a}$$

$$\beta = \frac{n\pi}{b}$$

$$\hat{A}_{ij} = A_{ij} - c_1 D_{ij}$$

$$\hat{B}_{ij} = B_{ij} - c_1 E_{ij}$$

$$\hat{D}_{ij} = D_{ij} - c_1 F_{ij}$$

$$\hat{F}_{ij} = F_{ij} - c_1 H_{ij}$$

$$\overline{A}_{ij} = A_{ij} - 2c_1 D_{ij} + c_1^{\,2} F_{ij}$$

$$\overline{D}_{ij} = D_{ij} - 2c_1 F_{ij} + c_1^{\,2} H_{ij} \quad (15)$$

4. RESULTS AND DISCUSSION

The FGM plate considered here consists of aluminium and alumina. The top surface is ceramic rich and the bottom surface is metal rich. The Young's modulus for alumina and aluminium are *Ec*=380 GPa and *Em*=70 GPa, respectively. The Poisson's ratio is chosen as constant, ν=0.3. The plate is of uniform thickness, and simply supported on all four edges. The maximum central deflections and stress values are determined using different homogenization schemes. The maximum central deflections and the stress values obtained using the Power law distribution function Mori-Tanaka homogenization scheme and exponential distribution function are compared with the values obtained by Ashraf M Zenkour.

TABLE 1 Comparison of obtained values by different homogenization methods with the values obtained by Ashraf M Zenkour(2004).

	w	σx	σy	τxy
Ashraf M Zenkour	0.5889	3.087	1.4894	0.611
P-FGM	0.58895	3.08501	1.4898	0.61111
M-FGM	0.59461	3.12506	1.42452	0.60655
E-FGM	0.62055	3.21721	1.4015	0.57686

*Reference[1]

The values of P-FGM are nearer to the reference values than the M-FGM and E-FGM.

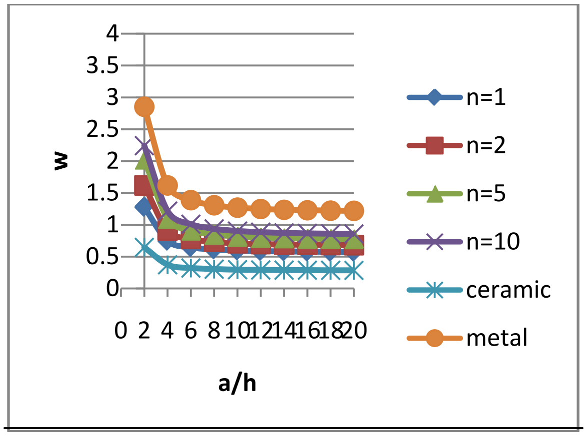

Fig. 5 Dimensionless center deflection (w) as a function of side to thickness ratio (a/h) of an FGM square plate

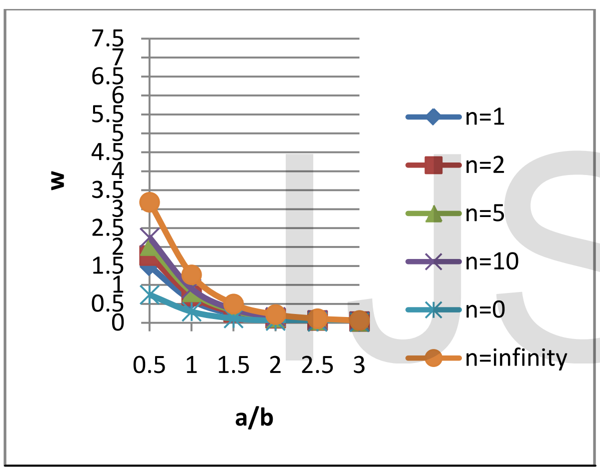

Fig. 6 Dimensionless center deflection (w) as a function of aspect ratio (a/b) of an FGM plate

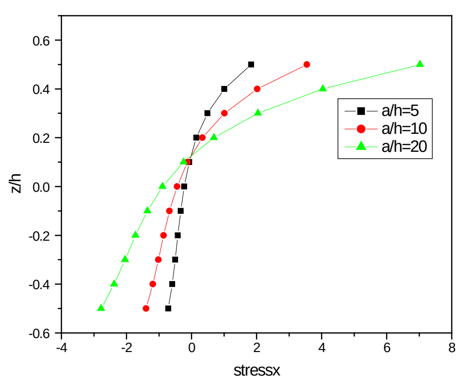

Fig. 7 Variation of longitudinal in-plane stress in X direction through the thickness

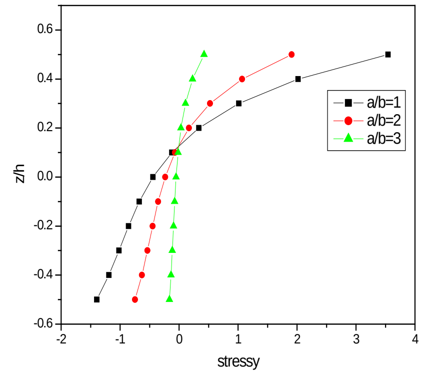

Fig. 8 Variation of longitudinal in-plane stress in Y direction through the thickness direction

The figure 7 and 8 show the through the thickness distributions of normal stresses in x and y directions respectively. The volume fraction index is taken as n=2. Also from the figure it is clear that the normal stresses σ_x and σy are compressive throughout the plate up to z/h =0.15 and then it become tensile. The maximum compressive stresses occur at a point on the bottom surface and maximum tensile stresses occur at a point on the top surface of the plate.

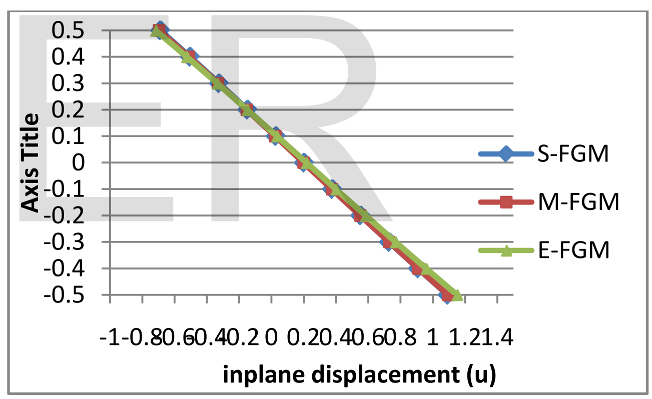

Fig. 9 Variation of in plane displacement (u) through thickness of an FGM plate subjected to sinusoidal loading (a/h=10)

5. CONCLUSION

The flexural behaviour of functionally graded material plate is analyzed based on Reddy's Higher Order Shear Deformation Theory. Different methods of material property homogenization are used and the results are compared with the exact solution. All the numerical results are obtained with the help of MATLAB programs. The effect of side to thickness ratio and aspect ratio on deflections and stresses are observed and compared the obtained values when different homogenization methods are used with the exact solutions. The main conclusions are

- The deflections become higher with increase in volume fraction *n*
- It is observed that the deflection and stress values of a functionally graded plate are intermediate to those of

metallic and ceramic plate.

- The deflections and stress values obtained by the power law distribution function and Mori-Tanaka homogenization scheme are observed to be closer than the values obtained by exponential distribution function.
- The better results are obtained by the higher order theory since it relaxes the assumption on the straightness and normality of a transverse normal after deformation by expanding the displacements as cubic functions of thickness coordinate.
- The deflections and stress values of P-FGM are found to be giving the closer results than the M-FGM and E-FGM.

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